

# Matrix Elements of the Linearized Collision Operator for Multi-temperature Gas-mixtures

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For a multi-component and multi-temperature gas-mixture the matrix elements of the linearized Boltzmann collision operator are investigated for isotropic interaction potentials. The representation by means of Burnett basis functions simplifies the algebraic structure and enables closed expressions for the general results, which can also be used for an investigation of inelastic collisions. For the elastic case those collision terms are given explicitly which appear in the balance equations for mass, momentum, energy and heat flux-vector.

## 1. Introduction

In the kinetic theory of gases, gas-mixtures and plasmas the velocity distribution functions for the components of the mixture under consideration are governed by the “kinetic equations”. In general this is a system of integro-differential equations coupled by the collision integral operators. The assumption that three- and more-particle collisions may be neglected yields a bilinear form of this operators [1]. A special example is the Boltzmann collision integral whose bilinear structure has been discussed in a previous paper [2].

Algebraic representations of the operators can be obtained explicitly by means of “basis systems” of functions. A suitable choice of this system facilitates the calculations. In the case of isotropic interaction potentials, e.g., the matrix of the linearized collision operator shows certain properties of diagonality, if basis functions are used which contain an irreducible representation of the group or rotations [3]. Special basis functions of that kind are the “Burnett functions”, which are products of Sonine (i.e. generalized Laguerre) functions and spherical harmonics. These spherical harmonics, written in a scalar notation, enable the use of group theoretical methods [4], well-known from quantum mechanics [5].

General calculations of the matrix elements of the linearized Boltzmann operator have been elaborated for Burnett functions by Aisbett et al. [6] for mixtures with uniform temperatures. Their results can be generalized for multitemperature mixtures [7], i.e. each component of the gas-

mixture has a kinetic temperature of its own. In particular for plasmas, where the temperatures of the electrons and heavy ions or neutrals often are quite different, the convergence of the moment-expansions can be improved by this technique.

In Sect. 2 of this paper we discuss the algebraic formalism, introducing as “generalized moments” the expansion coefficients of the distribution function. Their space- and time-dependence is governed by a system of “balance equations”. The representation of the collision operator consists of “collision elements”, coupling the generalized moments. Starting with the general case of the non-linearized collision operator its linearization is introduced. Then the collision elements are reduced to elements of matrices.

In Sect. 3 Burnett functions are introduced as a special basis system. The general results are given in Sect. 4 and specialized in Sect. 5 corresponding to Grad’s thirteen moment method, but without the restriction to elastic collisions. The explicit calculations have been carried out in a report [8], which is referred to in this paper in the form ( $R, \dots$ ).

The simplifications, which can be obtained after the assumption of elastic collisions, are discussed in Sect. 6.

## 2. General Formalism

Let  $f(i)$  be the distribution function for particle species  $i$ , which is governed by the kinetic equation

$$D(i)[f(i)] = \sum_j B(i, j)[f(i), f(j)]. \quad (2.1)$$

The operator  $D$ , which acts as a total time derivative, is not considered in this paper. The collision operators on the right-hand side of Eq. (2.1) may

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be assumed to be bilinear in the functions  $f(i)$  and  $f(j)$ . It has been shown [9] that Eq. (2.1) can be represented in its algebraical form

$$D_{\lambda\lambda'}(i) a_i^\lambda = \sum_j B_{\lambda\mu}^{\lambda'}(i, j) a_i^\lambda a_j^\mu, \quad (2.2)$$

where we use the convention to sum over all indices appearing once and only once as a superscript and a subscript. The quantities of Eq. (2.2) are defined by the introduction of a system of basis functions  $\varphi_\lambda(i)$  and their dual functions  $\varphi^\lambda(i)$ , which are determined by the duality relation

$$\langle \varphi^{\lambda'}(i) | \varphi_\lambda(i) \rangle = \delta_{\lambda\lambda'}. \quad (2.3)$$

If the procedure of algebraization takes into account only the velocity dependence of Eq. (2.1), the greek indices  $\lambda, \lambda'$  and  $\mu$  denote triples of indices, and the inner product symbol stands for the integration over the velocity space of particles  $i$ . The “generalized moments” in Eq. (2.2) are defined as

$$a_i^\lambda := \langle \varphi^\lambda(i) | f(i) \rangle \quad (2.4)$$

and the “collision elements” as

$$B_{\lambda\mu}^{\lambda'}(i, j) := \langle \varphi^{\lambda'}(i) | B(i, j) [\varphi_\lambda(i), \varphi_\mu(j)] \rangle. \quad (2.5)$$

The application of transformation theory allows to represent these collision elements as a linear combination [10],

$$B_{\lambda\mu}^{\lambda'}(i, j) = \Delta_{\lambda\mu, \kappa}^{\lambda', \kappa} \Psi_{\kappa}^{\kappa'}. \quad (2.6)$$

The conservation of the center-of-mass velocity during a binary collision causes  $\kappa$  to denote a triple of indices, related to basis functions depending on the relative velocity. The “collision matrix”  $\Psi$  depends on the interaction potential. The special case of isotropic interaction is considered in this paper, more details are given in the next section. The “combined transformation coefficients”  $\Delta$  of Eq. (2.6) are determined by the choice of the basis functions, which will be specialized in the next section. Here we must consider the linearizing procedure. Let the basis system be chosen such that  $\varphi_0(i)$  is proportional to the equilibrium distribution function and let us assume small deviations from this (partial local thermal) equilibrium. Hence we have

$$|a_{i,j}^\lambda| \ll |a_{i,j}^0| \quad \text{for } \lambda \geq 1. \quad (2.7)$$

Let  $\lambda, \mu \geq 1$ . Neglecting all quadratic terms with  $\lambda$  and  $\mu$  Eq. (2.2) leads to

$$D_0^{\lambda'}(i) a_i^0 + D_{\lambda\lambda'}^{\lambda'}(i) a_i^\lambda = B^{\lambda'}(i) \quad (2.8a)$$

with

$$B^{\lambda'}(i) := \sum_j \{ B_{00}^{\lambda'}(i, j) a_i^0 a_j^0 + B_{\lambda 0}^{\lambda'}(i, j) a_i^\lambda a_j^0 + B_{0\mu}^{\lambda'}(i, j) a_i^0 a_j^\mu \}. \quad (2.8b)$$

Therefore in the linearized theory we only need two kinds of  $\Delta$ -coefficients:  $\Delta_{0\mu, \kappa}^{\lambda', \kappa}$  and  $\Delta_{\lambda 0, \kappa}^{\lambda', \kappa}$  with  $0 \equiv (0, 0, 0)$ . The next section will show how this specialization facilitates our calculations.

### 3. Introduction of a Special Basis System

As mentioned above, the index  $\kappa$  of the collision matrices  $\Psi$  denotes triples of indices in the case of conserved center-of-mass velocity. We have discussed [11] the special case of conserved relative angular momentum, which means isotropic interaction. In this case it is advantageous to use a set of functions containing a representation of the rotation group. In our special form of basis functions we use the spherical harmonics  $Y_{lm}(\vartheta, \varphi)$  depending on the solid angle  $(\vartheta, \varphi)$  of some vector argument  $\mathbf{x}$ , which will be specified later. These spherical harmonics, most familiar as eigenfunctions of the quantum mechanical operators of the squared angular momentum and its projection onto the polar axis, are combined with a complete set of “radial” functions, depending on the absolute value of the vector argument. To define the special form of the basis functions we introduce at first a quantity  $\alpha$ , which normalizes the vector  $\mathbf{x}$  with respect to its physical dimension:

$$\mathbf{x} =: \alpha \check{\mathbf{x}}, \quad (3.1)$$

where now  $\check{\mathbf{x}}$  is a dimensionless vector. Moreover we use for abbreviation the variable

$$\varrho := \frac{1}{2} \check{\mathbf{x}}^2 \quad (3.2)$$

and define

$$\varphi_\lambda(\mathbf{x}) \equiv \Phi_{nlm}(\mathbf{x}) := \Phi_{nl}(\alpha, \varrho) Y_{lm}(\vartheta, \varphi) \quad (3.3)$$

with

$$\Phi_{nl}(\alpha, \varrho) := \alpha^{-(n+3)} (-1)^{\frac{1}{2}(n-l)} 2^{\frac{1}{2}(l+1)} \cdot \frac{1}{2\pi} \frac{1}{(n+l+1)!!} \cdot \sqrt{2l+1} \varrho^{l/2} L_{\frac{1}{2}(n-l)}^{(l+\frac{1}{2})}(\varrho) e^{-\varrho}. \quad (3.4)$$

As the generalized Laguerre polynomials [12] (or Sonine polynomials)

$$L_r^{(\beta)}(\varrho) := \sum_{k=0}^r \frac{(-\varrho)^k}{k!} \binom{r+\beta}{k+\beta} \quad (3.5)$$

are normalized by [13]

$$\int_0^\infty d\rho \rho^\beta L_r^{(\beta)}(\rho) L_{r'}^{(\beta)}(\rho) e^{-\rho} = \delta_{r,r'} \frac{\Gamma(\beta + r + 1)}{r!}, \quad (3.6)$$

we can evaluate the dual basis functions (2.3). The inner product

$$\langle | \rangle := \int d^3x \dots \quad (3.7)$$

immediately leads to

$$\Phi^{nlm}(\mathbf{x}) = \Phi_l^n(\alpha, \rho) Y_{lm}^*(\vartheta, \varphi) \quad (3.8)$$

with

$$\Phi_l^n(\alpha, \rho) = \alpha^n (-1)^{\frac{1}{2}(n-l)} 2^{\frac{1}{2}(l+1)} (n-l)!! \cdot \sqrt{\frac{2\pi}{2l+1}} \rho^{l/2} L_{\frac{1}{2}(n-l)}^{(l+\frac{1}{2})}(\rho). \quad (3.9)$$

All these equations require the following inequalities for the indices  $n, l$  and  $m$ :

$$l \stackrel{(2)}{\leq} n, \quad (3.10)$$

which means

$$l = n, n-2, n-4, \dots, \begin{cases} 1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even,} \end{cases} \quad (3.11)$$

and

$$-l \leq m \leq l. \quad (3.12)$$

If the velocity  $\mathbf{c}_i$  of a particle of species  $i$  is shifted by a mean velocity  $\mathbf{c}_0$ , which is a function of space and time, the “peculiar velocity”

$$\mathbf{C}_i := \mathbf{c}_i - \mathbf{c}_0 \quad (3.13)$$

is defined. Let

$$\mathbf{x} \equiv \mathbf{C}_i \quad (3.14)$$

and

$$\alpha \equiv \sqrt{k_B T_i / m_i} =: \gamma_i, \quad (3.15)$$

$\gamma_i^2$  being the mean-square thermal velocity of particle species  $i$  in a given direction [14]. Thus the functions  $\Phi_{nlm}(\mathbf{C}_i)$  and  $\Phi^{nlm}(\mathbf{C}_i)$  form the system of “one-particle basis functions” used to represent the one-particle distribution function. The related generalized moments (2.4),

$$a_i^{nlm} = \langle \Phi^{nlm}(\mathbf{C}_i) | f(i) \rangle \quad (3.16)$$

are discussed in the appendix A 1.

Furthermore we consider the relative velocity

$$\mathbf{g} := \mathbf{c}_j - \mathbf{c}_i = \mathbf{C}_j - \mathbf{C}_i \quad (3.17)$$

and the center-of-mass velocity

$$\begin{aligned} \mathbf{G} &:= \frac{m_i}{m_i + m_j} \mathbf{c}_i + \frac{m_j}{m_i + m_j} \mathbf{c}_j \\ &= \frac{m_i}{m_i + m_j} \mathbf{C}_i + \frac{m_j}{m_i + m_j} \mathbf{C}_j + \mathbf{c}_0. \end{aligned} \quad (3.18)$$

For these velocities it is useful to put

$$(\mathbf{x}, \alpha) \equiv (\mathbf{G}, \Gamma) \quad (3.19)$$

and

$$(\mathbf{x}, \alpha) \equiv (\mathbf{g}, \gamma) \quad (3.20)$$

respectively. The normalization quantities

$$\gamma := (\gamma_i^2 + \gamma_j^2)^{1/2} \quad (3.21)$$

and

$$\Gamma := (\gamma_i^{-2} + \gamma_j^{-2})^{-1/2} \quad (3.22)$$

are certain mean values of the thermal speeds  $\gamma_i$  and  $\gamma_j$ . It is convenient to use the basis functions  $\Phi_{qlm}(\mathbf{g})$ ,  $\Phi^{qlm}(\mathbf{g})$ ,  $\Phi_{QLM}(\mathbf{G})$  and  $\Phi^{QLM}(\mathbf{G})$  to represent the collision matrix  $\Psi$ , which in the special case of the Boltzmann collision integral with isotropic interaction takes the form [15]

$$\Psi_{q'lm}^{q'l'm'} = \delta_m^{m'} \delta_l^{l'} \Psi_q^{q'}(l). \quad (3.23)$$

The combined transformation coefficients  $\Delta$ , belonging to the collision matrices (3.23), can be given in two different forms [16]:

$$\begin{aligned} \Delta_{n_1 l_1 m_1, n_2 l_2 m_2; q'}^{n_1' l_1' m_1', q}(l) &= \sum_{QLM} \sum_m T_{QLM, q'lm}^{n_1' l_1' m_1', 000} T_{n_1 l_1 m_1, n_2 l_2 m_2}^{QLM, qlm} \\ &= \sum_m \iint d^3g d^3g' \Phi^{qlm}(\mathbf{g}') \\ &\quad \cdot J_{n_1 l_1 m_1, n_2 l_2 m_2}^{n_1' l_1' m_1'}(\mathbf{g}, \mathbf{g}') \Phi_{q'lm}(\mathbf{g}) \end{aligned} \quad (3.24)$$

with the “transformation coefficients” [17]

$$\begin{aligned} T_{QLM, q'lm}^{n_1' l_1' m_1', 000} &= \iint d^3g d^3G \Phi^{n_1' l_1' m_1'}(\mathbf{C}_i) \\ &\quad \cdot \Phi_{QLM}(\mathbf{G}) \Phi_{q'lm}(\mathbf{g}), \end{aligned} \quad (3.25)$$

$$\begin{aligned} T_{n_1 l_1 m_1, n_2 l_2 m_2}^{QLM, qlm} &= \iint d^3c_i d^3c_j \Phi^{QLM}(\mathbf{G}) \\ &\quad \cdot \Phi^{qlm}(\mathbf{g}) \Phi_{n_1 l_1 m_1}(\mathbf{C}_i) \\ &\quad \cdot \Phi_{n_2 l_2 m_2}(\mathbf{C}_j) \end{aligned} \quad (3.26)$$

and the functions [18]

$$\begin{aligned} J_{n_1 l_1 m_1, n_2 l_2 m_2}^{n_1' l_1' m_1'} &= \int d^3G \Phi^{n_1' l_1' m_1'}(\mathbf{C}_i) \\ &\quad \cdot \Phi_{n_1 l_1 m_1}(\mathbf{C}_i') \Phi_{n_2 l_2 m_2}(\mathbf{C}_j'). \end{aligned} \quad (3.27)$$

As was shown in Sect. 2, for the linearized theory we need the  $\Delta$ -coefficients either with  $n_1 = l_1 = m_1 = 0$  (denoted by an index  $h = 1$ ) or with  $n_2 = l_2 = m_2 = 0$  (denoted by  $h = 2$ ).

The peculiar velocities  $\mathbf{C}_i$  and  $\mathbf{C}_j$  as well as the center-of-mass and the relative velocities  $\mathbf{G}$  and  $\mathbf{g}$  can be used to define a "total angular momentum" operator

$$\mathbf{L} := \frac{1}{\sqrt{-1}} \left( \mathbf{g} \times \frac{\partial}{\partial \mathbf{g}} + \mathbf{G} \times \frac{\partial}{\partial \mathbf{G}} \right) \\ = \frac{1}{\sqrt{-1}} \left( \mathbf{C}_i \times \frac{\partial}{\partial \mathbf{C}_i} + \mathbf{C}_j \times \frac{\partial}{\partial \mathbf{C}_j} \right). \quad (3.28)$$

As the spherical harmonics  $Y_{lm}(\vartheta, \varphi)$ , used in the definition (3.3) of the basis functions, are eigenfunctions of the projection onto the polar-axis and the square of the constituents of the "total angular momentum operator" (3.28), we can apply the mechanism of coupling angular momenta, as used in quantum mechanics. This coupling scheme in connection with the Wigner-Eckert theorem can be used to prove (R, Sect. 2.6) that the  $\Delta$ -coefficients, Equation (3.24), are diagonal with respect to  $l_1', l_1$  and  $m_1', m_1$  in the case  $h=2$  and  $l_1', l_2$  and  $m_1', m_2$  in the case  $h=1$ . Moreover they are independent of  $m_1'$  and  $m_1, m_2$  respectively. Thus we are able to define "reduced  $\Delta$ -coefficients" by

$$\Delta_{n,q}^{n',q}(l', l) := \sum_{m'} \Delta_{000, n'l'm', q'}^{n'l'm', q}(l) \quad \text{for } h=1 \quad (3.29)$$

and

$$\Delta_{n,q}^{n',q}(l', l) := \sum_{m'} \Delta_{n'l'm', 000, q'}^{n'l'm', q}(l) \quad \text{for } h=2. \quad (3.30)$$

We retain the original  $\Delta$ -coefficients for the linearized case by

$$\Delta_{000, n_2 l_2' m_2', q'}^{n_1 l_1' m_1', q}(l) = \delta_{l_2' m_2'}^{l_1' m_1'} \frac{1}{2l_1' + 1} \cdot \Delta_{n_2, q'}^{n_1, q}(l_1', l), \quad (3.31)$$

and

$$\Delta_{n_1 l_1 m_1, 000; q'}^{n_1' l_1' m_1', q}(l) = \delta_{l_1' m_1'}^{l_1' m_1'} \frac{1}{2l_1' + 1} \cdot \Delta_{n_1, q'}^{n_1', q}(l_1', l). \quad (3.32)$$

#### 4. Calculation of the Reduced $\Delta$ -Coefficients

The calculation of the reduced  $\Delta$ -coefficients is very involved, therefore it will only be sketched here briefly (R, Sect. 3). In a first step the non-reduced  $\Delta$ -coefficients are calculated for  $n=l'=m'=0$  by means of the transformation coefficients  $T$ , Eqs. (3.25), (3.26), for  $l'=m'=0$  and  $n_1=l_1=m_1=n_2=l_2=m_2=0$ . The two types of transformation matrices are calculated in different ways. Straight-forward integration yields (R, Sect. 3.5)

$$T_{000, 000}^{QLM, qlm} = \delta^{M, -m} \delta^{L, l} (-1)^m \frac{1}{2l+1} \quad (4.1) \\ \cdot \tau^Q \gamma^{q+Q} \frac{(Q-l)!! (Q+l+1)!!}{(Q-q)!!}$$

with the parameter

$$\tau := \frac{1}{\gamma^2} \cdot \frac{k_B(T_j - T_i)}{m_i + m_j} \quad (4.2)$$

characterizing the difference of the temperatures of particle species  $i$  and  $j$ , normalized such that

$$|\tau| < 1. \quad (4.3)$$

The transformation coefficients of the other type can be obtained by means of recurrence relations. The result is (R, Sect. 3.4)

$$T_{QLM, q'lm}^{n'00, 000} = \delta_{M, -m} \delta_{L, l} (-1)^m (2l+1) \left( -\frac{m_j}{m_i + m_j} \right)^{q'} (\Gamma^2 + \omega^2 - \gamma_i^2)^{\frac{1}{2}(n'-Q-q')} \\ \cdot \frac{(n'+1)!}{(n'-Q-q')!! (Q+l+1)!! (Q-l)!! (q'+l+1)!! (q'-l)!!} \quad (4.4)$$

with

$$\omega := \gamma m_j / (m_i + m_j). \quad (4.5)$$

To calculate the corresponding reduced  $\Delta$ -coefficient (3.24) we multiply the  $T$ -coefficients, Eqs. (4.1) and (4.4), sum over  $Q, L, M$  and  $m$  and obtain (R, Sect. 3.6)

$$\Delta_{000, 000, q'}^{n'00, q}(l) = (-1)^{q'} (2l+1) \left( \frac{m_j}{m_i + m_j} \right)^{\frac{1}{2}(n'+q'-q)} \\ \cdot (\tau \gamma^2)^{\frac{1}{2}(n'+q-q')} \frac{1}{(\frac{1}{2}(n'-q-q'))!} \frac{(n'+1)!}{(q'+l+1)!! (q'-l)!!}. \quad (4.6)$$



The further steps of our calculation are based on the representation of the  $\Delta$ -coefficients by means of the  $J$ -functions, Equations (3.27). Using the algebraic properties of our basis functions we obtain two types of recurrence formulas. The first type contains differentiation operators like the Laplacian and  $\mathbf{C} \cdot \partial/\partial \mathbf{C}$ . It enables a shift of the index  $n$  by 2, i.e. if the reduced  $\Delta$ -coefficients are known for  $n$  and  $n-2$ , they can be evaluated for  $n+2$ . The second type of formulas is generated by multiplying the square of  $\mathbf{C}_i - \mathbf{C}_{i'}$  and  $\mathbf{C}_i - \mathbf{C}_{j'}$ , expressed in terms of  $\mathbf{g}$  and  $\mathbf{g}'$ , to the  $J$ -functions. The

resulting recurrence relations, which are very lengthly, enable the shift of the  $l'$ -index by 1, i.e. with known values of the reduced  $\Delta$ -coefficients for  $l' = n'$  and  $l' - 1 = n' - 1$  we can evaluate the coefficients for  $l' + 1 = n' + 1$ . Thus we can take the result (4.6), which is identical with the reduced  $\Delta$ -coefficient (3.29), (3.30), as initial value for the recurrence procedure of the second type.

For the last step of our calculation we use the recurrence procedure of the second type to evaluate the general form of the reduced  $\Delta$ -coefficients, which are given by [R, Eq. (3.8.44)]

$$\begin{aligned} \Delta_{h, q'}^{n', q}(l', l) &= (2l + 1)(2l' + 1)(\gamma^2 \tau)^{\frac{1}{2}(n' - n + q - q')} \\ &\cdot \frac{(n' + l' + 1)!! (n' - l')!!}{(q' + l + 1)!! (q' - l)!! (n + l' + 1)!! (n - l')!!} \Delta_{h, q'}^{n', q}(l', l) \end{aligned} \quad (4.7)$$

with the  $\Delta$ -coefficients in their general form [R, Eq. (3.8.44)]

$$\Delta_{1, q'}^{n', q}(l', l) = (-1)^{q'} \left( \frac{m_j}{m_i + m_j} \right)^{\frac{1}{2}(n' + n - q + q')} \sum_{k=0}^n \frac{1}{\left( \frac{n' - n - q - q'}{2} + k \right)!} \Gamma_n^{k, q}(l', l) \quad (4.8)$$

and

$$\begin{aligned} \Delta_{2, q'}^{n', q}(l', l) &= (-1)^{q'} \left( \frac{m_j}{m_i + m_j} \right)^{\frac{1}{2}(n' - n - q + q')} \\ &\cdot \sum_{k=0}^n \left( -\frac{m_j}{m_i + m_j} \right)^k \left( \frac{m_i}{m_i + m_j} \right)^{l' - k} \frac{1}{\left( \frac{n' - n - q - q'}{2} + k \right)!} \Gamma_n^{k, q}(l', l). \end{aligned} \quad (4.9)$$

For the coefficients  $\Gamma$  we have [R, Eq. (3.8.43)]

$$\begin{aligned} \Gamma_n^{k, q}(l', l) &= \sum_{\tau=0}^{[k/2]} \sum_{\varrho=0}^{\frac{1}{2}(n-l')} (-1)^{\tau+\varrho} \binom{n-2\varrho}{n-k} \binom{q-2\tau}{q-k} \\ &\cdot (2k - 2\tau - 2\varrho - 1)!! \frac{(q+l+1)!!}{(q+l+1-2\tau)!!} \binom{\frac{1}{2}(q-l)}{\tau} \frac{(n+l'+1)!!}{(n+l'+1-2\varrho)!!} \binom{\frac{1}{2}(n-l')}{\varrho}. \end{aligned} \quad (4.10)$$

For practical use it has turned out to be advantageous not to use the formulas (4.8)–(4.10) but to start with  $n = l'$  and to perform the recurrence procedure of the first type explicitly. For that reason we give as a conclusion of this section the two recurrence-formulas for  $h=1$  and  $h=2$  respectively [R, Eqs. (3.3.24), (3.3.25)]:

$$\begin{aligned} \Delta_{1, q'}^{n', q}(l', l) &= \frac{m_j}{m_i + m_j} (n' - n + q - q') \Delta_{1, q'}^{n', q}(l', l) + (q + l + 1)(q - l) \\ &\cdot \Delta_{1, q'}^{n', q-2}(l', l) - \Delta_{1, q'+2}^{n', q}(l', l) \end{aligned} \quad (4.11)$$

and

$$\begin{aligned} \Delta_{2, q'}^{n', q}(l', l) &= -\frac{m_i}{m_i + m_j} (n' - n + q - q') \Delta_{2, q'}^{n', q}(l', l) + (q + l + 1)(q - l) \\ &\cdot \Delta_{2, q'}^{n', q-2}(l', l) + \Delta_{2, q'}^{n'-2, q}(l', l) - \Delta_{2, q'+2}^{n', q}(l', l). \end{aligned} \quad (4.12)$$

## 5. Special Results

In this section we shall give some examples for the application of the formalism and of the results we have shown in the last section. Our aim is to write down the linearized collision terms for  $n' \leq 3$  and  $l' \leq 2$ . The index  $m'$  is irrelevant because of the independence of the matrix elements on  $m'$ . So we are left only with a pair of indices  $(n', l')$ , which facilitates the notation. The right-hand side of Eq. (2.8) can be re-written in terms of reduced  $\Delta$ -coefficients and  $\Psi$ -matrix elements:

$$B^{n'l'}(i) = \frac{1}{2l' + 1} \sum_j \sum_l \{ \delta_0^{l'} \Delta_{0,q}^{n',q}(0, l) a_i^{00} a_j^{00} + \sum_{n \geq 1} [\Delta_{n,q}^{n',q}(l', l) a_i^{n'l'} a_j^{00} + \Delta_{n,q}^{n',q}(l', l) a_i^{00} a_j^{n'l'}] \} \Psi_q^{q'}(l). \quad (5.1)$$

The index  $n'$  denotes the order of the generalized moment, the balance equation of which is considered, while the index  $l'$  is the according rank (cf. Appendix A 1). As tensorial moments of rank  $l' \geq 3$  are not directly measurable, we shall consider only the cases up to  $l' = 2$ .

For practical evaluations of the  $\Delta$ -coefficients it is quite useful to take into account some restriction rules for the indices (*R*, Sect. 4.2). Because of the structure of our basis functions we first have [cf. Eq. (3.10)]

$$l \leq^{(2)} q \quad (5.2)$$

and

$$l \leq^{(2)} q'. \quad (5.3)$$

This implies

$$|q' - q| \text{ even}. \quad (5.4)$$

Similarly we obtain

$$l' \leq^{(2)} n', \quad (5.5)$$

$$l' \leq^{(2)} n \quad (5.6)$$

and hence

$$|n' - n| \text{ even}. \quad (5.7)$$

Furthermore we have as restrictions for the indices  $q$  and  $q'$ :

$$0 \leq q' \leq n' \quad (5.8)$$

and

$$n - n' + q' \leq^{(2)} q \leq^{(2)} n + n' - q'. \quad (5.9)$$

A formula for the connection between  $l'$  and  $l$  can be deduced, which yields

$$l \leq \frac{1}{2} (n' + l'). \quad (5.10)$$

We now go back to expression (5.1). The reduced  $\Delta$ -coefficients, which are needed for the following results, are listed in the Appendix A 2.

To begin with  $n' = 0$ , implying  $l' = 0$ , we obtain the collision terms of the mass balance or continuity equation,

$$B^{00}(i) = \sum_j \{ \varrho_i \varrho_j \Psi_0^0(0) + \sum_{n \geq 2} \Psi_n^0(0) (\varrho_j a_i^{n0} + \varrho_i a_j^{n0}) \}, \quad (5.11)$$

where we used the mass density [cf. Eq. (2.4)]

$$a_{i,j}^{00} = \langle 1 | f(i, j) \rangle = \varrho_{i,j} \quad (5.12)$$

of the particles of species  $i, j$ . Note that the Expression (5.11) has no contributions in the elastic case, because the matrix  $\Psi$  then vanishes for  $l = 0$  [cf. Eq. (6.5)].

Next we consider the collision terms of the momentum balance ( $n' = 1, l' = 1$ ), the equation of motion:

$$B^{11}(i) = \sum_j \sum_{n \geq 1} \left\{ \varrho_j a_i^{n1} \left[ \frac{m_i}{m_i + m_j} \Psi_{n-1}^0(0) + \frac{m_j}{m_i + m_j} \Psi_n^1(1) - (n+1) \gamma^2 \tau \Psi_{n+1}^0(0) \right] + \varrho_i a_j^{n1} \left[ \frac{m_j}{m_i + m_j} (\Psi_{n-1}^0(0) - \Psi_n^1(1)) + (n+1) \gamma^2 \tau \Psi_{n+1}^0(0) \right] \right\}. \quad (5.13)$$

As demonstrated in the Appendix A 1, the index  $l' = 1$  describes the vector character of a quantity, especially we have the (spherical) components of the diffusion flux-vector for  $n' = 1$ ,

$$a_{i,j}^{11} \triangleq \langle \mathbf{C}_{i,j} | f(i, j) \rangle = \mathbf{J}_{i,j} \quad (5.14)$$

and of the heat flux-vector for  $n' = 3$ ,

$$a_{i,j}^{31} \triangleq \left\langle \mathbf{C}_{i,j} \left( \mathbf{C}_{i,j}^2 - 5 \frac{k_B T_{i,j}}{m_{i,j}} \right) | f(i, j) \right\rangle = 2 \mathbf{Q}_{i,j}. \quad (5.15)$$

Equation (5.13) describes the influence of  $\mathbf{J}$ ,  $\mathbf{Q}$  and higher order flux-vectors on the momentum balance.

There are two balance equations in the case  $n'=2$ . The first one ( $l'=0$ ) is the energy balance and the second one ( $l'=2$ ) is the balance for the stress tensor, which is necessary to evaluate the viscosity. The linearized collision terms are obtained by inserting the reduced  $\Delta$ -coefficients in (5.1). For  $l'=0$  we have

$$\begin{aligned}
 B^{20}(i) = \sum_j \left\{ \varrho_i \varrho_j \left[ 6 \gamma^2 \tau \frac{m_j}{m_i + m_j} (\Psi_n^0(0) - \Psi_{n+1}^1(1)) \right. \right. \\
 + \left( \frac{m_j}{m_i + m_j} \right)^2 \Psi_n^2(0) + 6 \gamma^4 \tau^2 \Psi_{n+2}^0(0) \Big] \\
 + \sum_{n \geq 2} \left( \varrho_i a_j^{n0} \left[ \left( \frac{m_j}{m_i + m_j} \right)^2 \right. \right. \\
 \cdot (\Psi_n^2(0) - 2 \Psi_{n-1}^1(1) + \Psi_{n-2}^0(0)) \\
 + 2(n+3) \gamma^2 \tau \frac{m_j}{m_i + m_j} \\
 \cdot (\Psi_n^0(0) - \Psi_{n+1}^0(1)) \\
 + (n+2)(n+3) \gamma^4 \tau^2 \Psi_{n+2}^0(0) \Big] \\
 + \varrho_j a_i^{n0} \left[ \left( \frac{m_j}{m_i + m_j} \right)^2 \Psi_n^2(0) \right. \\
 + 2 \frac{m_i m_j}{(m_i + m_j)^2} \Psi_{n-1}^1(1) \\
 + \left( \frac{m_i}{m_i + m_j} \right)^2 \Psi_{n-2}^0(0) \\
 + 2 \gamma^2 \tau \left( \left( 3 - (n+3) \frac{m_i}{m_i + m_j} \right) \Psi_n^0(0) \right. \\
 - (n+3) \frac{m_j}{m_i + m_j} \Psi_{n+1}^1(1) \Big) \\
 \left. \left. + (n+2)(n+3) \gamma^4 \tau^2 \Psi_{n+2}^0(0) \right] \right\}. \quad (5.16)
 \end{aligned}$$

Note that we have [cf. Eq. (A1.27)]

$$a_{i,j}^{20} = 3 \left( p_{i,j} - \frac{k_B T_{i,j}}{m_{i,j}} \varrho_{i,j} \right), \quad (5.17)$$

which means that  $a_{i,j}^{20}$  vanishes, if the ideal gas law is valid.

For  $l'=2$  we obtain

$$\begin{aligned}
 B^{22}(i) = \sum_j \sum_{n \geq 2} \left\{ \varrho_i a_j^{n2} \left[ \left( \frac{m_j}{m_i + m_j} \right)^2 (\Psi_{n-2}^0(0) \right. \right. \\
 \left. \left. - 2 \Psi_{n-1}^1(1) + \Psi_n^2(2)) \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & + 2n \gamma^2 \tau (\Psi_n^0(0) - \Psi_{n+1}^1(1)) \\
 & \left. + n(n+2) \gamma^4 \tau^2 \Psi_{n+2}^0(0) \right] \\
 & + \varrho_j a_i^{n2} \left[ \left( \frac{m_i}{m_i + m_j} \right)^2 \Psi_{n-2}^0(0) \right. \\
 & + 2 \frac{m_i m_j}{(m_i + m_j)^2} \Psi_{n-1}^1(1) \\
 & + \left( \frac{m_j}{m_i + m_j} \right)^2 \Psi_n^2(2) \\
 & - 2n \gamma^2 \tau \left( \frac{m_i}{m_i + m_j} \Psi_n^0(0) \right. \\
 & \left. + \frac{m_j}{m_i + m_j} \Psi_{n+1}^1(1) \right) \\
 & \left. \left. + n(n+2) \gamma^4 \tau^2 \Psi_{n+2}^0(0) \right] \right\}. \quad (5.18)
 \end{aligned}$$

As demonstrated in the Appendix A1, the index  $l'=2$  describes a quantity, which consists of the (spherical) components of a symmetric and traceless second rank tensor. In the case  $n=2$  we have

$$\begin{aligned}
 a_{i,j}^{22} & \triangleq \langle (\mathbf{C}_{i,j} \mathbf{C}_{i,j} - \frac{1}{3} C_{i,j}^2 \vec{I}) | f(i,j) \rangle \\
 & =: \vec{p}_{i,j} - p_{i,j} \vec{I} =: \vec{p}_{i,j}^0. \quad (5.19)
 \end{aligned}$$

We finish our list of special results by evaluating the collision term of the balance equation for the heat flux vector ( $n'=3$ ,  $l'=1$ ), needed for the heat conductivity [ $R$ , Eqs. (4.3.7) and (4.3.8)]:

$$\begin{aligned}
 B^{31}(i) = \sum_j \sum_{n \geq 1} \left\{ \varrho_i a_j^{n1} \left[ \left( \frac{m_j}{m_i + m_j} \right)^3 (\Psi_{n-3}^0(0) \right. \right. \\
 - 3 \Psi_{n-2}^1(1) + \frac{5}{3} \Psi_{n-1}^2(0) \\
 + \frac{4}{3} \Psi_{n-1}^2(2) - \Psi_n^3(1) \Big) \\
 + \gamma^2 \tau \left( \frac{m_j}{m_i + m_j} \right)^2 \left( (3n+7) \Psi_{n-1}^0(0) \right. \\
 - 2(3n+7) \Psi_n^1(1) \\
 + \frac{5}{3} (n+1) \Psi_{n+1}^2(0) \\
 + \frac{4}{3} (n+4) \Psi_{n+1}^2(2) \Big) \\
 + 3(n+4)(n+1) \gamma^4 \tau^2 \frac{m_j}{m_i + m_j} \\
 \cdot (\Psi_{n+1}^0(0) - \Psi_{n+2}^1(1)) \\
 + (n+4)(n+3)(n+1) \\
 \left. \left. \cdot \gamma^6 \tau^3 \Psi_{n+3}^0(0) \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
& + \varrho_j a_i^{n1} \left[ \left( \frac{m_i}{m_i + m_j} \right)^3 \Psi_{n-3}^0(0) \right. \\
& + 3 \frac{m_i^2 m_j}{(m_i + m_j)^3} \Psi_{n-2}^1(1) \\
& + \frac{m_j^2 m_i}{(m_i + m_j)^2} \left( \frac{5}{3} \Psi_{n-1}^2(0) \right. \\
& + \frac{4}{3} \Psi_{n-1}^2(2) \left. \right) + \left( \frac{m_j}{m_i + m_j} \right)^3 \Psi_{n-3}^3(1) \\
& + \gamma^2 \tau \left( \frac{m_i}{m_i + m_j} \left( 10 \frac{m_j}{m_i + m_j} \right. \right. \\
& - 3(n-1) \frac{m_i}{m_i + m_j} \left. \left. \right) \Psi_{n-1}^0(0) \right. \\
& + \frac{m_j}{m_i + m_j} \left( 10 \frac{m_j}{m_i + m_j} \right. \\
& - (6n+4) \frac{m_i}{m_i + m_j} \left. \right) \Psi_n^1(1) \\
& - \left( \frac{m_j}{m_i + m_j} \right)^2 \left( \frac{5}{3} (n+1) \Psi_{n+1}^2(0) \right. \\
& + \frac{4}{3} (n+4) \Psi_{n+1}^2(2) \left. \right) \left. \right) \\
& + (n+1) \gamma^4 \tau^2 \left( \left( (3n+2) \frac{m_i}{m_i + m_j} \right. \right. \\
& - 10 \frac{m_j}{m_i + m_j} \left. \left. \right) \Psi_{n+1}^0(0) \right. \\
& + 3(n+4) \frac{m_j}{m_i + m_j} \Psi_{n+2}^1(1) \left. \right) \\
& - (n+4)(n+3)(n+1) \\
& \cdot \gamma^6 \tau^3 \Psi_{n+3}^0(0) \left. \right] \left. \right\}. \quad (5.20)
\end{aligned}$$

The terms with  $n=1$  express the influence of the diffusion fluxes on the heat flux balance. Here we must add the remark [cf. Eqs. (5.2), (5.3)]

$$\Psi_q^{q'}(l) = 0 \quad \text{if } q', q < l. \quad (5.21)$$

These special collision terms shall do as examples, they are sufficient for a reproduction of the thirteen moment method. It must be mentioned, however, that up to now the assumption of elastic collision was not yet made. The simplifications, generated by this assumption, are discussed in the next section.

## 6. Elastic Collisions

The  $\Psi$ -matrix in Eq. (2.6) can be represented as [19]

$$\begin{aligned}
\Psi_q^{q'}(l) = & -4\pi \int_0^\infty g^3 dg \Phi_l^{q'}(\gamma, g) \\
& \cdot \left\{ \sigma^0(g) \Phi_{ql}(\gamma, g) - \sigma^l(g) \frac{1}{2l+1} \right. \\
& \left. \cdot \Phi_{ql}(\gamma, g') \right\}, \quad (6.1)
\end{aligned}$$

where the  $\Phi$ -functions are the “radial” parts of our basis functions, given in Eqs. (3.4) and (3.9). The  $g$ -dependent coefficients  $\sigma^l(g)$  are defined as expansion coefficients of the differential cross section for isotropic interaction,

$$\sigma_{ij}(g, \chi) =: \sum_l \sigma^l(g) P_l(\cos \chi), \quad (6.2)$$

which depends on the angle of deflection

$$\chi := \arccos \hat{\mathbf{g}} \cdot \hat{\mathbf{g}}'. \quad (6.3)$$

In the case of elastic collisions we have  $g=g'$ , and with

$$\sigma^l(g) = \frac{2l+1}{2} \int d\cos \chi \sigma_{ij}(g, \chi) P_l(\cos \chi) \quad (6.4)$$

we can write Eq. (6.1) in the form

$$\begin{aligned}
\Psi_q^{q'}(l) = & -2\pi \int_0^\infty g^3 dg \Phi_l^{q'}(\gamma, g) \Phi_{ql}(\gamma, g) \\
& \cdot \int d\cos \chi \sigma_{ij}(g, \chi) (1 - P_l(\cos \chi)). \quad (6.5)
\end{aligned}$$

We define the “transfer collision frequencies”

$$\begin{aligned}
\nu_{ij}^{(l)}(g) := & 2\pi g \frac{\varrho_j}{m_j} \\
& \cdot \int d\cos \chi \sigma_{ij}(g, \chi) (1 - P_l(\cos \chi)) \quad (6.6)
\end{aligned}$$

and re-write Eq. (6.5) as

$$\begin{aligned}
\Psi_q^{q'}(l) = & -\frac{m_j}{\varrho_j} \\
& \cdot \int_0^\infty g^2 dg \Phi_l^{q'}(\gamma, g) \Phi_{ql}(\gamma, g) \nu_{ij}^{(l)}(g). \quad (6.7)
\end{aligned}$$

We substitute the  $\Phi$ -functions by means of the Eqs. (3.4) and (3.9) and obtain

$$\begin{aligned}
\Psi_q^{q'}(l) = & -\gamma^{q'-q} (-1)^{\frac{1}{2}(q'+q)+l} 2^{l+3/2} \frac{1}{\sqrt{2\pi}} \\
& \cdot \frac{(q'-l)!!}{(q+l+1)!!} \frac{m_j}{\varrho_j} \\
& \cdot \int_0^\infty d\varepsilon \varepsilon^{l+1/2} L_{\frac{1}{2}(q'-l)}^{(l+1/2)}(\varepsilon) L_{\frac{1}{2}(q-l)}^{(l+1/2)}(\varepsilon) e^{-\varepsilon} \nu_{ij}^{(l)}(g) \quad (6.8)
\end{aligned}$$



with

$$\varepsilon := \frac{1}{2} (g/\gamma)^2. \quad (6.9)$$

We couple the two Laguerre polynomials,

$$L_{\frac{1}{2}(q-l)}^{(l+1/2)}(\varepsilon) L_{\frac{1}{2}(q'-l)}^{(l+1/2)}(\varepsilon) = \sum_s a_q^{sq'}(l) L_s^{(l+1/2)}(\varepsilon), \quad (6.10)$$

and obtain

$$\begin{aligned} \Psi_q^{q'}(l) &= (-1)^{\frac{1}{2}(q'+q)+l+1} \gamma^{q'-q} \frac{m_j}{\varrho_j} \frac{(q'-l)!!}{(q+l+1)!!} \\ &\cdot \sum_s \frac{(2s+2l+1)!!}{(2s)!!} (-1)^s a_q^{sq'}(l) v^{(ls)} \end{aligned} \quad (6.11)$$

with the “transport collision frequencies” [20]

$$\begin{aligned} v^{(ls)} &:= (-1)^s \frac{s!}{(s+l+\frac{1}{2})!} \\ &\cdot \int_0^\infty d\varepsilon \varepsilon^{l+1/2} L_s^{(l+1/2)}(\varepsilon) e^{-\varepsilon} v_{ij}^{(l)}(g), \end{aligned} \quad (6.12)$$

which vanish for all  $s \geq 1$  in the case of Maxwell interaction.

Though the coupling coefficients  $a$  in Eq. (6.10) can be evaluated in general [21], we shall give here only some of their properties. Equation (6.10) yields

$$\left| \frac{q-q'}{2} \right| \leq s \leq \frac{q+q'}{2} - l, \quad (6.13)$$

$$a_q^{sq'}(l) = a_q^{sq'}(l) \quad (6.14)$$

and

$$a_q^{sl}(l) = \delta_{\frac{1}{2}(q-l)}^s. \quad (6.15)$$

One more property is needed for our special results,

$$\begin{aligned} a_q^{sl+2}(l) &= \frac{q+l+1}{2} \delta_{\frac{1}{2}(q-l-2)}^s - (q-l) \delta_{\frac{1}{2}(q-l)}^s \\ &+ \frac{q-l+2}{2} \delta_{\frac{1}{2}(q-l+2)}^s. \end{aligned} \quad (6.16a)$$

It can be deduced expressing

$$L_1^{(l+1/2)}(\varepsilon) L_{\frac{1}{2}(q-l)}^{(l+1/2)}(\varepsilon) = (l + \frac{3}{2} - \varepsilon) L_{\frac{1}{2}(q-l)}^{(l+1/2)}(\varepsilon) \quad (6.16b)$$

in terms of Laguerre polynomials [22]. So we obtain for the  $\Psi$ -matrices (6.11)

$$\Psi_q^l(l) = -\gamma^{l-q} \frac{m_j}{\varrho_j} \frac{1}{(q-l)!!} v^{[l(q-l)/2]} \quad (6.17)$$

and with Eq. (6.16a)

$$\begin{aligned} \Psi_q^{l+2}(l) &= -\gamma^{l-q+2} \frac{m_j}{\varrho_j} \frac{1}{(q-l)!!} \\ &\cdot \{ (q-l) v^{[l(q-l-2)/2]} + 2(q-l) v^{[l(q-l)/2]} \\ &+ (q+l+3) v^{[l(q-l+2)/2]} \}. \end{aligned} \quad (6.18)$$

Before inserting the Eqs. (6.17) and (6.18) in the special results of Sect. 5 we note that because of  $P_0 \equiv 1$  Eq. (6.5) obviously implies

$$\Psi_q^{q'}(0) = 0. \quad (6.19)$$

Hence we obtain in the case of elastic collisions the following linearized collision terms: The right-hand side  $B^{00}(i)$  of the mass-balance equation (5.11) vanishes. For  $n'=1, l'=1$  we have from Eq. (5.13)

$$\begin{aligned} B^{11}(i) &= - \sum_j \sum_{n \geq 1} \frac{m_j}{m_i + m_j} \frac{1}{\gamma^{n-1}} \frac{1}{(n-1)!!} \frac{m_j}{\varrho_j} \\ &\cdot v^{(1[n-1]/2)} (\varrho_j a_i^{n1} - \varrho_i a_j^{n1}). \end{aligned} \quad (6.20)$$

For  $n'=2, l'=0$  we obtain from Eq. (5.16)

$$\begin{aligned} B^{20}(i) &= \sum_j \frac{m_j}{\varrho_j} \left\{ 6\gamma^2 \tau \frac{m_j}{m_i + m_j} \varrho_i \varrho_j v^{(10)} \right. \\ &+ \sum_{n \geq 2} 2 \frac{m_j}{m_i + m_j} \gamma^{2-n} \frac{1}{n!!} \left( \varrho_i a_j^{n0} \right. \\ &\cdot \left[ n \frac{m_j}{m_i + m_j} v^{(1[n-2]/2)} + (n+3) \tau v^{(1[n/2])} \right] \\ &- \varrho_j a_i^{n0} \left[ n \frac{m_i}{m_i + m_j} v^{(1[n-2]/2)} \right. \\ &\left. \left. \left. - (n+3) \tau v^{(1[n/2])} \right] \right] \right\}. \end{aligned} \quad (6.21)$$

For  $n'=2, l'=2$  Eq. (5.18) yields

$$\begin{aligned} B^{22}(i) &= \sum_j \frac{m_j}{\varrho_j} \sum_{n \geq 2} \frac{m_j}{m_i + m_j} \gamma^{2-n} \frac{1}{(n-2)!!} \\ &\cdot \left\{ \varrho_i a_j^{n2} \left[ \frac{m_j}{m_i + m_j} (2 v^{(1[n-2]/2)} - v^{(2[n-2]/2)}) \right. \right. \\ &+ 2 \tau v^{(1[n/2])} \left. \right] - \varrho_j a_i^{n2} \left[ 2 \frac{m_i}{m_i + m_j} v^{(1[n-2]/2)} \right. \\ &\left. \left. + \frac{m_j}{m_i + m_j} v^{(2[n-2]/2)} - 2 \tau v^{(1[n/2])} \right] \right\}. \end{aligned} \quad (6.22)$$

We have in the case  $n'=3, l'=1$  from Eq. (5.20)

$$\begin{aligned} B^{31}(i) &= \sum_j \frac{m_j}{\varrho_j} \sum_{n \geq 1} \frac{m_j}{m_i + m_j} \gamma^{3-n} \frac{1}{(n-1)!!} \\ &\cdot \left\{ \varrho_i a_j^{n1} \left[ \left( \frac{m_j}{m_i + m_j} \right)^2 (4(n-1) v^{(1[n-3]/2)} \right. \right. \\ &- \frac{4}{3} (n-1) v^{(2[n-3]/2)} \\ &+ 2(n-1) v^{(1[n-1]/2)} + (n+4) v^{(1[n+1]/2)}) \\ &+ \tau \frac{m_j}{m_i + m_j} \left( (6n+14) v^{(1[n-1]/2)} \right. \\ &\left. \left. - \frac{4}{3} (n+4) v^{(2[n-1]/2)} \right) \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + 3(n+4) \tau^2 \nu^{(1[n+1]/2)} \Bigg] \\
& - \varrho_j a_i^{n1} \left[ \frac{3m_i^2 + m_j^2}{(m_i + m_j)^2} (n-1) \nu^{(1[n-3]/2)} \right. \\
& + \frac{4}{3} (n-1) \frac{m_i m_j}{(m_i + m_j)^2} \nu^{(2[n-3]/2)} \\
& + \left( \frac{m_j}{m_i + m_j} \right)^2 (2(n-1)) \nu^{(1[n-1]/2)} \\
& + (n+4) \nu^{(1[n+1]/2)} \\
& + \tau \left( \frac{10m_j - (6n+4)m_i}{m_i + m_j} \nu^{(1[n-1]/2)} \right. \\
& - \frac{4}{3} (n+4) \frac{m_j}{m_i + m_j} \nu^{(2[n-1]/2)} \Bigg) \\
& \left. + 3(n+4) \tau^2 \nu^{(1[n+1]/2)} \right] \Bigg\}. \quad (6.23)
\end{aligned}$$

The linearized collision terms, which we have given in the Eqs. (6.20)–(6.23), show that the evaluation of special collision terms is a matter of straight-forward calculation, once the general formalism has been established. If the system of balance equations is to be truncated at the level  $n' = 3$ ,  $l' = 1$ , the matrix elements of the linearized collision operator are needed only for  $n' \leq 3$ , too. The higher terms with  $n' \geq 4$  can be used to estimate the influence of the higher moments in the balance equations under consideration.

## 7. Conclusion

The evaluation of the elements of the collision matrix for isotropic interaction potentials, which has been demonstrated in this paper, is based on the knowledge of the set of transport collision frequencies, which have been investigated in the case of elastic collisions. They are combined linearly with the generalized moments by means of the reduced  $\Delta$ -coefficients, which play the most important role in this paper. Though explicit results have been deduced only for elastic collisions (Sect. 6), we can expect that the  $\Delta$ -coefficients maintain their importance in the inelastic case (cf. Section 5).

One of the basic physical assumptions is that the system under consideration is weakly deviated from local partial thermal equilibrium, which means — among others — that we are dealing with different temperatures for different components of the gasmixture or plasma. The natural result is the

parameter  $\tau$ , which appears in the formulas for the reduced  $\Delta$ -coefficients. Provided both temperatures are given this multi-temperature concept ensures a better convergence of the basic series expansions of the distribution functions. On the other hand this concept requires a lot of a priori information, because — in any case — the existence of different temperatures is the result of external forces as in plasma physics and so the temperatures would have to be determined in a self-consistent way. This problem, however, was not to be discussed in this paper.

Once the matrix elements of the linearized collision operator are known for given interaction potentials, we have to look after a suitable method to solve the linearized algebraic form of the kinetic equation. The well-known method in this context is to truncate the balance equations by physical arguments and to solve the residual system. It would be interesting, however, to elaborate a technique, which does not need such an a priori truncation. In each case the first step for further calculations is the representation of the differential operator, the evaluation of the “left-hand side” of the kinetic equation. This will be the subject of a subsequent paper.

At last we should mention that there might be a competition between higher order corrections in the solution of the linearized problem and effects, which arise from the non-linearity of the kinetic equation [23]. The answer, however, to this question depends on the special physical situation and must be given self-consistently.

## Appendix

### A 1. Generalized Moments Related to the Burnett Basis Functions

The interpretation of the generalized moments

$$a_i^{nlm} = \langle \Phi^{nlm}(\mathbf{C}_i) | f(i) \rangle \equiv a_i^{nl} \quad (A1)$$

of the distribution function  $f(i)$  provides a brief introduction of spherical tensor components: Consider a cartesian frame of reference,  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ . We define “complex basis vectors”  $\chi_m$  by

$$\begin{aligned}
\chi_{-1} &:= \frac{1}{\sqrt{2}} (\mathbf{e}_1 - i \mathbf{e}_2), \\
\chi_0 &:= \mathbf{e}_3, \\
\chi_{+1} &:= -\frac{1}{\sqrt{2}} (\mathbf{e}_1 + i \mathbf{e}_2),
\end{aligned} \quad (A2)$$

which yields the dual complex basis

$$\mathbf{x}^m = \mathbf{x}_m^* . \quad (\text{A3})$$

By means of the vectors (A2) and (A3) we construct  $l$ -th order completely symmetric and traceless “spherical basis tensors”

$$\Xi_{lm} := \alpha_{lm} \overline{\mathbf{x}_{m_1} \mathbf{x}_{m_2} \dots \mathbf{x}_{m_l}} =: (\Xi_{lm})^* \quad (\text{A4})$$

with

$$m = \sum_{j=1}^l m_j , \quad (\text{A5})$$

where  $\alpha_{lm}$  is a normalizing factor, which is determined by the normalization condition

$$\Xi_{lm'}^{(i)} \Xi_{lm} = \delta_m^{m'} . \quad (\text{A6})$$

The symbol  $\overline{\phantom{x}}$  characterizes the result of the symmetrizing procedure, which is extended by  $\overline{\phantom{x}}^0$  to completely symmetric and traceless tensors.

Representing a given vector  $\mathbf{x}$  by its polar coordinates  $r, \vartheta, \varphi$  we obtain for the  $l$ -th order tensorial product of the vector  $\mathbf{x}$  ( $R$ , Sect. A2.5)

$$\mathbf{x}^l_{(i)} \Xi_{lm} = \alpha_l r^l Y_{lm}(\vartheta, \varphi) . \quad (\text{A7})$$

The definition (A4) can be extended to “completely symmetric basis tensors of order  $n$  and rank  $l$ ” by the recursive definition

$$\Xi_{lm}^{(n)} := \Xi_{lm} \quad \text{for } n = l , \quad (\text{A8})$$

$$\Xi_{lm}^{(n+2)} := \alpha_{nlm} \overline{\Xi_{lm}^{(2)}} I , \quad (\text{A9})$$

where again the normalizing factors  $\alpha_{nlm}$  are obtained by

$$\Xi_{l'm'}^{(n)} \Xi_{lm}^{(n)} = \delta_l^{l'} \delta_m^{m'} . \quad (\text{A10})$$

By combinatorial considerations it can be shown ( $R$ , Sect. A2.4) that these basis tensors form a complete set in the subspace of completely symmetric tensors with

$$n \geq l \quad (\text{A11})$$

and

$$-l \leq m \leq +l . \quad (\text{A12})$$

Let  $\overset{(n)}{a}$  be an  $n$ -th order completely symmetric tensor. It can be decomposed into its “irreducible” parts:

$$\overset{(n)}{a} = \overset{(n)}{a}^{nlm} \Xi_{lm} = \overset{(n)}{a}_{nlm} \Xi^{lm} , \quad (\text{A13})$$

where the coefficients

$$\overset{(n)}{a}^{nlm} = \overset{(n)}{a}_{(n)}^{nlm} \Xi_{lm} \quad (\text{A14})$$

and

$$\overset{(n)}{a}_{nlm} = \overset{(n)}{a}_{(n)}^{nlm} \Xi_{lm} \quad (\text{A15})$$

are called the “spherical (contra- or covariant resp.) components” of the tensor  $\overset{(n)}{a}$ . Because of Eq. (A7) the set of  $2l+1$  spherical components of a tensor of rank  $l$  transforms like the set of spherical harmonics  $Y_{lm}(\vartheta, \varphi)$  under the group of rotations.

Now we can go back to the generalized moments (A1). Up to  $(n', l') = (3, 1)$  we have [cf. Eqs. (3.8), (3.9)]

$$\Phi^{000}(C_i) \equiv 1 , \quad (\text{A16})$$

$$\Phi^{11m}(C_i) = \sqrt{\frac{4\pi}{3}} C_i Y_{1m}^*(\vartheta_i, \varphi_i) , \quad (\text{A17})$$

$$\Phi^{200}(C_i) = C_i^2 - 3 \frac{k_B T_i}{m_i} , \quad (\text{A18})$$

$$\Phi^{22m}(C_i) = \sqrt{\frac{4\pi}{5}} C_i^2 Y_{2m}^*(\vartheta_i, \varphi_i) , \quad (\text{A19})$$

and

$$\begin{aligned} \Phi^{31m}(C_i) = & \sqrt{\frac{4\pi}{3}} C_i \left( C_i^2 - 5 \frac{k_B T_i}{m_i} \right) \\ & \cdot Y_{1m}^*(\vartheta_i, \varphi_i) . \end{aligned} \quad (\text{A20})$$

The inner product of (A16) with  $f(i)$  yields the normalization of  $f(i)$ ,

$$\overset{(0)}{a}_i^{00} \equiv \overset{(0)}{a}_i^{000} = \langle 1 | f(i) \rangle =: \varrho_i \quad (\text{A21})$$

with the mass density  $\varrho_i$  of particle species  $i$ . Instead of Eq. (A17) we can write

$$\Phi^{11m}(C_i) = C_i \cdot \overset{(1)}{\Xi}^{1m} , \quad (\text{A22})$$

hence the inner product  $\overset{(1)}{a}_i^{11} \equiv \overset{(1)}{a}_i^{11m}$  yields the spherical components of the diffusion flux-vector

$$\mathbf{J}_i := \langle C_i | f(i) \rangle . \quad (\text{A23})$$

We define the pressure tensor by

$$\vec{\vec{p}}_i := \langle C_i C_i | f(i) \rangle \quad (\text{A24})$$

and write

$$\vec{\vec{p}}_i =: \vec{\vec{p}}_i^0 + p_i \overset{(2)}{I} \quad (\text{A25})$$

with the traceless stress tensor  $\vec{\vec{p}}_i^0$  and the scalar pressure  $p_i$ . Hence we have

$$p_i = \frac{1}{3} \langle C_i^2 | f(i) \rangle , \quad (\text{A26})$$

which means, in connection with Eq. (A18):

$$a_i^{20} \equiv a_i^{200} = 3(p_i - k_B T_i/m_i). \quad (\text{A27})$$

So the validity of the ideal gas law, which can be used to define the temperatures, causes a vanishing right-hand side of Equation (A27). Instead of Eq. (A19) we can write

$$\Phi^{22m}(C_i) = C_i C_i : \Xi^{2m}, \quad (\text{A28})$$

which means that the inner products  $a_i^{22} \equiv a_i^{22m}$  are just the spherical components of the stress tensor  $\vec{p}_i^0$ .

Equation (A20) can be written in the form

$$\Phi^{31m}(C_i) = \left( C_i^2 - 5 \frac{k_B T_i}{m_i} \right) C_i : \Xi^{1m}, \quad (\text{A29})$$

hence the inner products again are spherical components of a vector, in this case the heat flux-vector

$$Q_i := \frac{1}{2} I : \langle C_i C_i C_i | f(i) \rangle - \frac{5}{2} \frac{k_B T_i}{m_i} J_i \quad (\text{A30})$$

and we have

$$a_i^{31} \equiv a_i^{31m} = 2 Q_i : \Xi^{1m}. \quad (\text{A31})$$

Alternatively we can write because of the completeness of the  $\Xi_{lm}^{(n)}$ :

$$a_i^{31} \equiv a_i^{31m} \quad (\text{A32}) \\ = \alpha \left\{ \langle C_i C_i C_i | f(i) \rangle - 3 \frac{k_B T_i}{m_i} J_i I \right\} : \Xi^{1m},$$

where the constant  $\alpha$  is unimportant in this context. This demonstrates that the generalized moments  $a_i^{nl} \equiv a_i^{nlm}$  are the contravariant spherical components of a tensor of order  $n$  and rank  $l$ .

## A2. Reduced $\Delta$ -Coefficients for $n' \leq 3, l' \leq 2$

In this section we give a list of the non-vanishing reduced  $\Delta$ -coefficients, which are needed for the special results of Sects. 5 and 6. Their evaluation has been discussed in Section 4.

$n' = 0, l' = 0$ :

$$\Delta_{n,0}^{0,0}(0,0) = 1;$$

$n' = 1, l' = 1$ :

$$\Delta_{n,0}^{1,n+1}(1,0) = (-1)^{h+1} \cdot 3(n+1) \gamma^2 \tau,$$

$$\Delta_{n,0}^{1,n-1}(1,0) = 3 \frac{m_{j,i}}{m_i + m_j} \text{ for } h = 1, 2,$$

$$\Delta_{n,1}^{1,n}(1,1) = (-1)^h \cdot 3 \frac{m_j}{m_i + m_j};$$

$n' = 2, l' = 0$ :

$$\Delta_{n,0}^{2,n+2}(0,0) = (n+2)(n+3)(\gamma^2 \tau)^2,$$

$$\Delta_{n,0}^{2,n}(0,0) = 2\gamma^2 \tau \begin{cases} (n+3) \frac{m_j}{m_i + m_j} & \text{for } h = 1, \\ 3 - (n+3) \frac{m_i}{m_i + m_j} & \text{for } h = 2, \end{cases}$$

$$\Delta_{n,0}^{2,n-2}(0,0) = \left( \frac{m_{j,i}}{m_i + m_j} \right)^2 \text{ for } h = 1, 2,$$

$$\Delta_{n,1}^{2,n+1}(0,1) = -2(n+3) \gamma^2 \tau \frac{m_j}{m_i + m_j},$$

$$\Delta_{n,1}^{2,n-1}(0,1) = (-1)^h \cdot 2 \frac{m_j m_{j,i}}{(m_i + m_j)^2} \text{ for } h = 1, 2,$$

$$\Delta_{n,2}^{2,n}(0,0) = \left( \frac{m_j}{m_i + m_j} \right)^2;$$

$n' = 2, l' = 2$ :

$$\Delta_{n,2}^{2,n+2}(2,0) = 5n(n+2)(\gamma^2 \tau)^2,$$

$$\Delta_{n,0}^{2,n}(2,0) = (-1)^{h+1} \cdot 10n \gamma^2 \tau \frac{m_{j,i}}{m_i + m_j} \text{ for } h = 1, 2,$$

$$\Delta_{n,0}^{2,n-2}(2,0) = 5 \left( \frac{m_{j,i}}{m_i + m_j} \right)^2 \text{ for } h = 1, 2,$$

$$\Delta_{n,1}^{2,n+1}(2,1) = -10n \gamma^2 \tau \frac{m_j}{m_i + m_j},$$

$$\Delta_{n,1}^{2,n-1}(2,1) = (-1)^h \cdot 10 \frac{m_j m_{j,i}}{(m_i + m_j)^2} \text{ for } h = 1, 2,$$

$$\Delta_{n,2}^{2,n}(2,2) = 5 \left( \frac{m_j}{m_i + m_j} \right)^2;$$

$n' = 3, l' = 1$ :

$$\Delta_{n,0}^{3,n+3}(1,0) = (-1)^{h+1} \cdot 3(n+4)(n+3) \cdot (n+1)(\gamma^2 \tau)^3,$$

$$\Delta_{n,0}^{3,n+1}(1,0) = 3(n+1)(\gamma^2 \tau)^2$$

$$\begin{cases} 3(n+4) \frac{m_j}{m_i + m_j} & \text{for } h = 1, \\ (3n+2) \frac{m_i}{m_i + m_j} - 10 \frac{m_j}{m_i + m_j} & \text{for } h = 2, \end{cases}$$



$$\begin{aligned}
\Delta_{h,n,0}^{3,n-1}(1,0) &= 3\gamma^2 \tau \\
&\cdot \begin{cases} (3n+7) \left( \frac{m_j}{m_i+m_j} \right)^2 & \text{for } h=1, \\ -3(n-1) \left( \frac{m_i}{m_i+m_j} \right)^2 + 10 \frac{m_i m_j}{(m_i+m_j)^2} & \text{for } h=2, \end{cases} \\
\Delta_{h,n,0}^{3,n-3}(1,0) &= 3 \left( \frac{m_{j,i}}{m_i+m_j} \right)^3 \text{ for } h=1,2, \\
\Delta_{h,n,1}^{3,n+2}(1,1) &= (-1)^h \cdot 9(\gamma^2 \tau)^2 (n+4) \\
&\cdot (n+1) \frac{m_j}{m_i+m_j}, \\
\Delta_{h,n,1}^{3,n}(1,1) &= -6\gamma^2 \tau \\
&\cdot \begin{cases} (3n+7) \left( \frac{m_j}{m_i+m_j} \right)^2 & \text{for } h=1 \\ (3n+2) \frac{m_i m_j}{(m_i+m_j)^2} - 5 \left( \frac{m_j}{m_i+m_j} \right)^2 & \text{for } h=2, \end{cases} \\
\Delta_{h,n,1}^{3,n-2}(1,1) &= (-1)^h \cdot 9 \frac{m_j m_{j,i}^2}{(m_i+m_j)^3} \text{ for } h=1,2, \\
\Delta_{h,n,2}^{3,n+1}(1,2) &= (-1)^{h+1} \cdot 4(n+4) \gamma^2 \tau \left( \frac{m_j}{m_i+m_j} \right)^2, \\
\Delta_{h,n,2}^{3,n+1}(1,0) &= (-1)^{h+1} \cdot 5(n+1) \gamma^2 \tau \left( \frac{m_j}{m_i+m_j} \right)^2, \\
\Delta_{h,n,2}^{3,n-1}(1,2) &= 4 \frac{m_j^2 m_{j,i}}{(m_i+m_j)^3} \text{ for } h=1,2, \\
\Delta_{h,n,2}^{3,n-1}(1,0) &= 5 \frac{m_j^2 m_{j,i}}{(m_i+m_j)^3} \text{ for } h=1,2, \\
\Delta_{h,n,3}^{3,n}(1,1) &= (-1)^h \cdot 3 \left( \frac{m_j}{m_i+m_j} \right)^3.
\end{aligned}$$

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